

ON EQUALITY OF CERTAIN AUTOMORPHISM GROUPS

SURJEET KOUR, VISHAKHA

ABSTRACT. Let $G = H \times A$ be a group, where H is a purely non-abelian subgroup of G and A is a non-trivial abelian factor of G . Then, for $n \geq 2$, we show that there exists an isomorphism $\phi : Aut_{Z(G)}^{\gamma_n(G)}(G) \rightarrow Aut_{Z(H)}^{\gamma_n(H)}(H)$ such that $\phi(Aut_c^{n-1}(G)) = Aut_c^{n-1}(H)$. Also, for a finite non-abelian p -group G satisfying a certain natural hypothesis, we give some necessary and sufficient conditions for $Autcent(G) = Aut_c^{n-1}(G)$. Furthermore, for a finite non-abelian p -group G we study the equality of $Autcent(G)$ with $Aut_{Z(G)}^{\gamma_n(G)}(G)$.

1. INTRODUCTION

Let G be a group with the center $Z(G)$ and the automorphism group $Aut(G)$. For an abelian group H , $Hom(G, H)$ denotes the abelian group of all homomorphisms from G to H . For a given group G along with some nice properties, it would be interesting to examine the properties of its automorphism group and to determine relations among various subgroups of its automorphism group. So it is natural to look for appropriate conditions on a group G such that one can describe its automorphism group “ $Aut(G)$ ” or could find out certain subgroups of $Aut(G)$ are isomorphic. Also, the group of homomorphisms is far better understood than the group of automorphisms and therefore it would be in a great interest of many mathematicians to see isomorphisms between subgroups of an automorphism group and suitable homomorphism groups. Recently, many mathematicians got interested to study the equality of certain subgroups of an automorphism group and a number of results have been proved in this direction. Many interesting results have been proved in [5], [8], [6], [9] etc..

Most of the results, which have been proved in this direction, are either for a finite group or for a finite non-abelian p -group of nilpotency class at most two. Our main motivation behind this work is to study relations among certain subgroups of the automorphism group of two groups (need not be finite) and to study the relations among certain subgroups of the automorphism group of a finite non-abelian p group. In Section 3 and 4, we obtain quite a few interesting results for infinite groups and for finite non-abelian p -groups. These results are generalizations of some results which were proved either for a finite group or for a finite non-abelian p -group of class at most two.

2010 *Mathematics Subject Classification.* 20D15, 20D45.

Key words and phrases. Finite group, p - group, Class preserving automorphism, Central automorphism.

An automorphism α of G is called central if $g^{-1}\alpha(g) \in Z(G)$ for all $g \in G$. The set of all central automorphisms, denoted by $Aut_{cent}(G)$, is a normal subgroup of $Aut(G)$. In [1], Adney and Yen have obtained a very interesting relation between the central automorphism group $Aut_{cent}(G)$ of a finite non-abelian group G and the homomorphism group $Hom(G, Z(G))$. They proved that, for a purely non-abelian finite group G (a finite non-abelian group G is said to be purely non-abelian, if it has no non-trivial abelian direct factor), there exists a bijection between $Aut_{cent}(G)$ and $Hom(G, Z(G))$. This result is being used extensively in the study of central automorphism group and it is also referred at various places in this article.

Let M and N be two normal subgroups of G . Let $Aut^M(G)$ denote the group of all automorphisms of G which fix M setwise and act trivially on G/M and let $Aut_N(G)$ denote the group of all automorphisms of G which fix N element wise. The group $Aut^M(G) \cap Aut_N(G)$ is denoted by $Aut_N^M(G)$. Note that $Aut_{cent}(G) = Aut_{\gamma_2(G)}^{Z(G)}(G)$.

Definition 1. An automorphism β of G is called n^{th} class preserving if for each $g \in G$, there exists $x \in \gamma_n(G)$ such that $\beta(g) = x^{-1}gx$, where $\gamma_n(G)$ denotes the n^{th} term of lower central series. We denote the group of all n^{th} class preserving automorphisms by $Aut_c^n(G)$.

Note that $Aut_c^1(G)$ is denoted by $Aut_c(G)$ and called the group of all class preserving automorphisms.

Let H_1, H_2, \dots, H_k be k groups and let G be the direct product of H_i 's. In Section 3, we obtain relations between subgroups of $Aut(G)$ and subgroups of $Aut(H_j)$, which fix certain subgroups and quotient groups of the group G and the group H_j . More precisely, in Theorem 3.1 we prove that for two normal subgroups M_j and N_j of H_j , if $M = \{1\} \times \dots \times M_j \times \dots \times \{1\}$ and $N = H_1 \times \dots \times N_j \times \dots \times H_k$, then $Aut_N^M(G)$ is isomorphic to $Aut_{N_j}^{M_j}(H_j)$.

As a consequence of Theorem 3.1, we obtain Theorem 3.2 which states that if $G = H \times A$ is a non-abelian group, where H is a purely non-abelian subgroup of G and A is a non-trivial abelian factor of G , then there exists an isomorphism $\phi : Aut_{Z(G)}^{\gamma_n(G)}(G) \rightarrow Aut_{Z(H)}^{\gamma_n(H)}(H)$ such that $\phi(Aut_c^{n-1}(G)) = Aut_c^{n-1}(H)$.

In [8, Theorem 4.1], Yadav proved that if G and H are two finite non-abelian isoclinic groups, then $Aut_c(G)$ is isomorphic to $Aut_c(H)$. Later on, Rai in [7, Theorem A] has extended Yadav's above result to the group $Aut_{Z(G)}^{\gamma_2(G)}(G)$, which states that if two finite non-abelian groups G and H are isoclinic, then there exists an isomorphism $\phi : Aut_{Z(G)}^{\gamma_2(G)}(G) \rightarrow Aut_{Z(H)}^{\gamma_2(H)}(H)$ such that $\phi(Aut_c(G)) = Aut_c(H)$.

When $G = H \times A$, Theorem A of Rai [7], the extension of Yadav [8, Theorem 4.1], can be obtained from Theorem 3.2 by taking G to be a finite group and $n = 2$.

In the same article [8], Yadav also proved that for a finite nilpotent group G of class two, the group of all class preserving automorphisms is isomorphic to a subgroup of $Hom(G/Z(G), \gamma_2(G))$. In Section 3, Theorem 3.4, we generalize this result and prove that if G is a finite nilpotent group of class at most n and H is a subgroup of G such that $\gamma_n(G) \subseteq H \subseteq Z(G)$, then the group of $(n-1)^{th}$ class preserving automorphism is isomorphic to a subgroup of $Hom(G/H, \gamma_n(G))$.

Yadav's above result can be obtained from Theorem 3.4 by taking $n = 2$ and $H = Z(G)$.

In Section 4, we obtain various equalities between the group of central automorphisms of a finite non-abelian p -group G and suitable subgroups of the automorphism group $\text{Aut}(G)$ by putting some conditions on the group G .

In [6, Theorem 3.1], Kalra and Gumber proved that for a finite p -group G , $\text{Autcent}(G) = \text{Aut}_c(G)$ if and only if $\text{Aut}_c(G)$ is isomorphic to $\text{Hom}(G/Z(G), \gamma_2(G))$ and $\gamma_2(G) = Z(G)$. In Theorem 4.1, we prove that for a finite non-abelian p -group G and for $n \geq 2$, if $\text{Autcent}(G) = \text{Aut}_c^{n-1}(G)$ then $\gamma_n(G) \leq Z(G)$ and $d(Z(G)) = d(\gamma_n(G))$ also if $\exp(G/\gamma_2(G)) > \text{var}(\gamma_n(G), Z(G))$ then $\gamma_n(G) = Z(G)$ and $\text{Aut}_c^{n-1}(G)$ is isomorphic to $\text{Hom}(G/Z(G), \gamma_n(G))$. We obtain Kalra and Gumber [6, Theorem 3.1] as Corollary 2 of Theorem 4.1.

In [7, Theorem B(1)], Rai also studied the equality of the group of central automorphisms with $\text{Aut}_{Z(G)}^{\gamma_2(G)}(G)$. He proved that for a finite non-abelian p -group G of nilpotency class two, $\text{Autcent}(G) = \text{Aut}_{Z(G)}^{\gamma_2(G)}(G)$ if and only if $\gamma_2(G) = Z(G)$. In Theorem 4.2, for $n \geq 2$, we prove that for a finite non-abelian p -group G , if $\text{Autcent}(G) = \text{Aut}_{Z(G)}^{\gamma_n(G)}(G)$ then $\gamma_n(G) \leq Z(G)$ and $d(Z(G)) = d(\gamma_n(G))$ also if $\exp(G/\gamma_2(G)) > \text{var}(\gamma_n(G), Z(G))$ then $\gamma_n(G) = Z(G)$, which generalizes Rai's above result.

2. NOTATIONS AND PRELIMINARIES

Throughout the article, n denotes a positive integer and p denotes a prime number. We use 1 to denote the identity of a group. For basic definitions we refer [4]. Here we recall some definitions:

The exponent of a group G , denoted by $\exp(G)$, is the smallest positive integer n such that $g^n = 1$ for all $g \in G$. For a finite abelian group G , $d(G)$ denotes the rank of G . Let G and H be two finite abelian p -groups. Let

$$G = C_{p^{n_1}} \times C_{p^{n_2}} \times \cdots \times C_{p^{n_s}},$$

$$H = C_{p^{m_1}} \times C_{p^{m_2}} \times \cdots \times C_{p^{m_t}}$$

be the cyclic decompositions of G and H respectively, where $n_i \geq n_{i+1}$ and $m_j \geq m_{j+1}$ are positive integers and C_{p^i} denotes the cyclic group of order p^i . If $d(G) = d(H)$, that is $s = t$, and G is a proper subgroup of H , then there exists a unique integer r such that $n_r < m_r$ and $n_j = m_j$ for all $r + 1 \leq j \leq t$. In this case, we define $\text{var}(G, H) = p^{n_r}$. In other words, $\text{var}(G, H)$ denotes the order of the last cyclic factor of G whose order is less than that of the corresponding cyclic factor of H . If $G = H$, then we say $\text{var}(G, H) = 1$.

The following lemmas are well known.

Lemma 1. *Let G be a nilpotent group of class n . Then $\exp(G/Z(G))$ divides $\exp(\gamma_n(G))$. Furthermore, if $n = 2$, then $\exp(G/Z(G)) = \exp(\gamma_2(G))$.*

Lemma 2. *Let G be a group and let H, K be two normal subgroups of G such that $H \subseteq K$. Then $\exp(G/K)$ divides $\exp(G/H)$.*

Lemma 3. *Let $G = H \times K$, where H and K are two groups. If K is abelian, then G is nilpotent of class n if and only if H is nilpotent of class n .*

For a purely non-abelian finite group G , Adney and Yen proved the following result.

Theorem 2.1. *[1, Theorem 1] Let G be a purely non-abelian finite group. Then $|Autcent(G)| = |Hom(G, Z(G))|$.*

Note that $Hom(G, Z(G))$ is isomorphic to $Hom(G/\gamma_i(G), Z(G))$ for all $i \geq 2$. So, for $i \geq 2$, we have

$$|Autcent(G)| = |Hom(G/\gamma_i(G), Z(G))|.$$

The following lemmas are proved in [3] and [5].

Lemma 4. *[3, Corollary 3.3] Let $G = H \times K$, where H and K have no common direct factor. Then*

$$|Autcent(G)| = |Autcent(H)||Autcent(K)||Hom(H, Z(K))||Hom(K, Z(H))|.$$

Lemma 5. *[5, Lemma 2.2(i)] Let G, H and K be finite abelian p -groups with $H \subseteq K$. Then the following statements are equivalent:*

- (1) $Hom(G, H) = Hom(G, K)$;
- (2) Either $H = K$ or $d(H) = d(K)$ and $exp(G) \leq var(H, K)$.

3. CLASS PRESERVING AUTOMORPHISMS

Let H_1, H_2, \dots, H_k be k groups (need not be finite) and let G be the direct product of H_i 's. Let π_j denote the projection of G on H_j . Here we prove some relations among the subgroups of $Aut(G)$ and $Aut(H_j)$, where $j \in \{1, \dots, k\}$.

Theorem 3.1. *Let H_1, H_2, \dots, H_k be k groups and let $G = H_1 \times H_2 \times \dots \times H_k$. For a fixed $j \in \{1, \dots, k\}$, let M_j and N_j be two normal subgroups of H_j . If $M = \{1\} \times \dots \times M_j \times \dots \times \{1\}$ and $N = H_1 \times \dots \times N_j \times \dots \times H_k$, then*

- (1) *For each $f \in Aut_N^M(G)$ the map $\alpha_f : H_j \rightarrow H_j$ given by $\alpha_f(h_j) = \pi_j of(1, \dots, h_j, \dots, 1)$ is an automorphism and it belongs to $Aut_{N_j}^{M_j}(H_j)$.*
- (2) *The map $\phi : Aut_N^M(G) \rightarrow Aut_{N_j}^{M_j}(H_j)$ given by $\phi(f) = \alpha_f$ is an isomorphism.*

Proof. (1) First we show that α_f is an automorphism. Clearly, α_f is a homomorphism as f and π_j are homomorphisms.

To prove α_f is injective, consider $h_j \in H_j$ such that $\alpha_f(h_j) = 1$. Note that $(1, \dots, h_j, \dots, 1) \in G$. Since $f \in Aut_N^M(G)$, therefore there exists $m_j \in M_j$ such that $f(1, \dots, h_j, \dots, 1) = (1, \dots, h_j m_j, \dots, 1)$. Then, $\alpha_f(h_j) = \pi_j(1, \dots, h_j m_j, \dots, 1) = h_j m_j = 1$. Thus $f(1, \dots, h_j, \dots, 1) = (1, \dots, 1)$. Since f is injective, therefore $h_j = 1$. Hence α_f is injective.

Now we show that α_f is surjective. Let $h_j \in H_j$. Since f is onto, therefore there exists $(a_1, \dots, a_k) \in G$ such that $f(a_1, \dots, a_k) = (1, \dots, h_j, \dots, 1)$. As f fixes N element wise, $f(1, \dots, a_j, \dots, 1) = (a_1^{-1}, \dots, h_j, \dots, a_k^{-1})$. Therefore, $\alpha_f(a_j) = \pi_j(a_1^{-1}, \dots, h_j, \dots, a_k^{-1}) = h_j$. Hence α_f is an automorphism.

Also $h_j^{-1}\alpha_f(h_j) \in M_j$ for all $h_j \in H_j$ and $\alpha_f(n_j) = n_j$ for all $n_j \in N_j$. Thus $\alpha_f \in \text{Aut}_{N_j}^{M_j}(H_j)$.

- (2) To prove that ϕ is a homomorphism, consider $f_1, f_2 \in \text{Aut}_N^M(G)$ and $h_j \in H_j$. Let $x_1, x_2 \in M_j$ be such that $f_2(1, \dots, h_j, \dots, 1) = (1, \dots, h_j x_2, \dots, 1)$ and $f_1 \circ f_2(1, \dots, h_j, \dots, 1) = f_1(1, \dots, h_j x_2, \dots, 1) = (1, \dots, h_j x_2 x_1, \dots, 1)$.

Then

$$\begin{aligned} \alpha_{f_1} \alpha_{f_2}(h_j) &= \alpha_{f_1}(\pi_j \circ f_2(1, \dots, h_j, \dots, 1)) = \alpha_{f_1}(h_j x_2) \\ &= \pi_j \circ f_1(1, \dots, h_j x_2, \dots, 1) = h_j x_2 x_1 \\ \Rightarrow \alpha_{f_1} \alpha_{f_2}(h_j) &= \pi_j \circ (f_1 \circ f_2)(1, \dots, h_j, \dots, 1) = \alpha_{f_1 \circ f_2}(h_j). \end{aligned}$$

Therefore, $\phi(f_1 \circ f_2) = \alpha_{f_1 \circ f_2} = \alpha_{f_1} \alpha_{f_2} = \phi(f_1) \phi(f_2)$.

To prove that ϕ is injective, consider $f \in \text{Aut}_N^M(G)$ such that $\phi(f) = 1$, that is, $\phi(f)(h_j) = \alpha_f(h_j) = h_j$ for all $h_j \in H_j$. Since $f \in \text{Aut}_N^M(G)$, therefore there exists $m_j \in M_j$ such that $f(1, \dots, h_j, \dots, 1) = (1, \dots, h_j m_j, \dots, 1)$. Now $h_j = \alpha_f(h_j) = \pi_j \circ f(1, \dots, h_j, \dots, 1) = h_j m_j$ and therefore $m_j = 1$. Thus $f(1, \dots, h_j, \dots, 1) = (1, \dots, h_j, \dots, 1)$ for all $h_j \in H_j$. As f fixes N element wise, $f(a_1, \dots, a_k) = (a_1, \dots, a_k)$ for all $(a_1, \dots, a_k) \in G$. Hence ϕ is injective.

Now we show that ϕ is surjective. Let $\psi \in \text{Aut}_{N_j}^{M_j}(H_j)$. Define $f_\psi : G \rightarrow G$ as $f_\psi(h_1, \dots, h_j, \dots, h_k) = (h_1, \dots, \psi(h_j), \dots, h_k)$. It is easy to observe that $f_\psi \in \text{Aut}_N^M(G)$. Furthermore, $\phi(f_\psi) = \psi$. Thus ϕ is an isomorphism. \square

In [7, Theorem A], Rai proved that if G and H are two finite isoclinic groups, then there exists an isomorphism $\Psi : \text{Aut}_{Z(G)}^{\gamma_2(G)}(G) \rightarrow \text{Aut}_{Z(H)}^{\gamma_2(H)}(H)$ such that $\Psi(\text{Aut}_c(G)) = \text{Aut}_c(H)$. Note that if $G = H \times A$, where H is a purely non-abelian subgroup of G and A is a non-trivial abelian factor of G , then G and H are isoclinic. As a consequence of Theorem 3.1, we obtain the following result which generalizes the above result of Rai [7, Theorem A] to any group (need not be finite) in the case $G = H \times A$.

Theorem 3.2. *Let $G = H \times A$ be a group, where H is a purely non-abelian subgroup of G and A is a non-trivial abelian factor of G . Then, for $n \geq 2$, there exists an isomorphism $\phi : \text{Aut}_{Z(G)}^{\gamma_n(G)}(G) \rightarrow \text{Aut}_{Z(H)}^{\gamma_n(H)}(H)$ such that $\phi(\text{Aut}_c^{n-1}(G)) = \text{Aut}_c^{n-1}(H)$.*

Proof. Note that $Z(G) = Z(H) \times A$ and $\gamma_n(G) = \gamma_n(H) \times \{1\}$. Then, by Theorem 3.1(2), we have an isomorphism $\phi : \text{Aut}_{Z(G)}^{\gamma_n(G)}(G) \rightarrow \text{Aut}_{Z(H)}^{\gamma_n(H)}(H)$ given by $\phi(f) = \alpha_f$, where α_f is the map defined in Theorem 3.1(1). Next, we show that $\phi(\text{Aut}_c^{n-1}(G)) = \text{Aut}_c^{n-1}(H)$. Let π_H be the projection of G on H . Consider $f \in \text{Aut}_c^{n-1}(G) \subseteq \text{Aut}_{Z(G)}^{\gamma_n(G)}(G)$, then $\phi(f) \in \text{Aut}_{Z(H)}^{\gamma_n(H)}(H)$. Also for any $h \in H$, we have $\phi(f)(h) = \alpha_f(h) = \pi_H(f(h, 1)) = k^{-1}hk$; where $f(h, 1) = (k^{-1}hk, 1)$ for some $k \in \gamma_{n-1}(H)$. Thus $\phi(f) \in \text{Aut}_c^{n-1}(H)$. Hence $\phi(\text{Aut}_c^{n-1}(G)) \subseteq \text{Aut}_c^{n-1}(H)$.

Now, consider $\psi \in \text{Aut}_c^{n-1}(H)$ and define $f_\psi : G \rightarrow G$ as $f_\psi(h, a) = (\psi(h), a)$ for all $(h, a) \in G$. It is easy to see that $f_\psi \in \text{Aut}_c^{n-1}(G)$. Also for $h \in H$, $\phi(f_\psi)(h) = \alpha_{f_\psi}(h) = \pi_H(f_\psi(h, 1)) = \pi_H(\psi(h), a) = \psi(h)$. Hence $\phi(f_\psi) = \psi$. Thus $\psi \in \phi(\text{Aut}_c^{n-1}(G))$. Hence $\text{Aut}_c^{n-1}(H) \subseteq \phi(\text{Aut}_c^{n-1}(G))$. \square

Lemma 6. *Let G be a group. Let M and N be two normal subgroups of G such that M is abelian and $[N, M] = 1$. Then the following statements are true:*

- (1) *For each $f \in \text{Aut}_N^M(G)$ the map $\alpha_f : G/N \rightarrow M$ given by $\alpha_f(gN) = g^{-1}f(g)$ is well defined.*
- (2) *If $f_1, f_2 \in \text{Aut}_N^M(G)$ and $f_1 \neq f_2$ then $\alpha_{f_1} \neq \alpha_{f_2}$.*

Proof. (1) Let $g_1, g_2 \in G$ with $g_1N = g_2N$. Then $g_1 = g_2n$ for some $n \in N$. Observe that

$$g_1^{-1}f(g_1) = (g_2n)^{-1}f(g_2n) = n^{-1}g_2^{-1}f(g_2)f(n) = n^{-1}g_2^{-1}f(g_2)n.$$

Since $g_2^{-1}f(g_2) \in M$ and $[N, M] = 1$, therefore $g_2^{-1}f(g_2)$ commutes with n . Thus $g_1^{-1}f(g_1) = g_2^{-1}f(g_2)$.

- (2) If $f_1, f_2 \in \text{Aut}_N^M(G)$ and $f_1 \neq f_2$. Then $f_1(g) \neq f_2(g)$ for some $g \in G$. This implies that $g^{-1}f_1(g) \neq g^{-1}f_2(g)$. Therefore $\alpha_{f_1} \neq \alpha_{f_2}$. \square

A subgroup H of a group G is called central if $H \subseteq Z(G)$.

Theorem 3.3. *Let G be a group and let M and N be two normal subgroups of G . Suppose, M is a central subgroup of G . Then the following statements are true:*

- (1) *For each $f \in \text{Aut}_N^M(G)$ the map $\alpha_f : G/N \rightarrow M$ given by $\alpha_f(gN) = g^{-1}f(g)$ is well defined and it is a homomorphism.*
- (2) *If G is finite and $M \subseteq N$, then the map $\phi : \text{Aut}_N^M(G) \rightarrow \text{Hom}(G/N, M)$ defined by $\phi(f) = \alpha_f$ is an isomorphism.*

Proof. (1) By Lemma 6(1) the map α_f is well defined. Let $g_1, g_2 \in G$. Then

$$\alpha_f(g_1Ng_2N) = \alpha_f(g_1g_2N) = g_2^{-1}g_1^{-1}f(g_1)f(g_2).$$

Since $g_1^{-1}f(g_1) \in M \subseteq Z(G)$, therefore $\alpha_f(g_1Ng_2N) = g_1^{-1}f(g_1)g_2^{-1}f(g_2) = \alpha_f(g_1N)\alpha_f(g_2N)$. Thus α_f is a homomorphism.

- (2) Let $f_1, f_2 \in \text{Aut}_N^M(G)$. We claim that $\alpha_{f_1}\alpha_{f_2} = \alpha_{f_1 \circ f_2}$. Let $g \in G$. Since $g^{-1}f_2(g) \in M \subseteq N$, therefore $gN = f_2(g)N$. Then

$$\begin{aligned} (\alpha_{f_1}\alpha_{f_2})(gN) &= \alpha_{f_1}(gN)\alpha_{f_2}(gN) \\ &= \alpha_{f_1}(f_2(g)N)\alpha_{f_2}(gN) \\ &= f_2(g^{-1})f_1(f_2(g))g^{-1}f_2(g). \end{aligned}$$

Since $g^{-1}f_2(g) \in M \subseteq Z(G)$, therefore $g^{-1}f_2(g)f_2(g^{-1})f_1(f_2(g)) = g^{-1}f_1(f_2(g))$ and hence $(\alpha_{f_1}\alpha_{f_2})(gN) = \alpha_{f_1 \circ f_2}(gN)$. This proves that the map ϕ is a homomorphism.

By Lemma 6(2) ϕ is injective. It remains to show that ϕ is surjective. Let $\psi \in \text{Hom}(G/N, M)$ be arbitrary. We define the map $f_\psi : G \rightarrow G$ as $f_\psi(g) = g\psi(gN)$ for all $g \in G$.

First we claim that f_ψ is a homomorphism. Let $g_1, g_2 \in G$. Then

$$f_\psi(g_1g_2) = g_1g_2\psi(g_1g_2N) = g_1g_2\psi(g_1N)\psi(g_2N).$$

Since $\psi(g_1N) \in M \subseteq Z(G)$, therefore $g_1g_2\psi(g_1N)\psi(g_2N) = g_1\psi(g_1N)g_2\psi(g_2N)$ and hence $f_\psi(g_1g_2) = f_\psi(g_1)f_\psi(g_2)$.

Next we claim that f_ψ is injective. Let $g \in G$ with $f_\psi(g) = 1$. Then $g\psi(gN) = 1$ and so $\psi(gN) = g^{-1}$. Since $\psi(gN) \in M \subseteq N$, it follows that $g^{-1} \in N$ and so $\psi(gN) = 1$. Thus $g = 1$.

Since the map f_ψ is injective and G is finite, therefore f_ψ is surjective. Thus $f_\psi \in \text{Aut}(G)$. Now it is easy to observe that $f_\psi \in \text{Aut}_N^M(G)$ and $\phi(f_\psi) = \psi$. Hence the map ϕ is surjective. \square

Let G be a nilpotent group of class at most n and H be a central subgroup of G . We denote the group $\{f \in \text{Hom}(G/H, \gamma_n(G)) \mid f(gH) \in [g, \gamma_{n-1}(G)] \forall g \in G\}$ by $\text{Hom}_c(G/H, \gamma_n(G))$.

In [8, Proposition 3.1], Yadav proved that for a finite nilpotent group G of class 2, $\text{Aut}_c(G)$ is isomorphic to $\text{Hom}_c(G/Z(G), \gamma_2(G))$. We generalize this result to a finite nilpotent group of class n . Also we observe that the result is true for any central subgroup H of G which contains $\gamma_n(G)$.

Theorem 3.4. *Let G be a finite nilpotent group of class at most n and let H be a central subgroup of G such that $\gamma_n(G) \leq H$. Then, for $n \geq 2$, $\text{Aut}_c^{n-1}(G)$ is isomorphic to $\text{Hom}_c(G/H, \gamma_n(G))$.*

Proof. Since $\gamma_n(G) \leq H$, therefore by Theorem 3.3(2) if we take $M = \gamma_n(G)$ and $N = H$, the map $\phi : \text{Aut}_H^{\gamma_n(G)}(G) \rightarrow \text{Hom}(G/H, \gamma_n(G))$ defined by $\phi(f) = \alpha_f$ is an isomorphism.

Note that $\text{Aut}_c^{n-1}(G) \subseteq \text{Aut}_H^{\gamma_n(G)}(G)$ and hence the restriction map $\phi : \text{Aut}_c^{n-1}(G) \rightarrow \text{Hom}(G/H, \gamma_n(G))$ is an injective homomorphism. It remains to show that $\phi(\text{Aut}_c^{n-1}(G)) = \text{Hom}_c(G/H, \gamma_n(G))$. Let $f \in \text{Aut}_c^{n-1}(G)$ be arbitrary. Then for a given $g \in G$, there exists $x \in \gamma_{n-1}(G)$ depending on g , such that $f(g) = x^{-1}gx$. Thus

$$\phi(f)(gH) = \alpha_f(gH) = g^{-1}f(g) = g^{-1}x^{-1}gx = [g, x] \in [g, \gamma_{n-1}(G)].$$

Therefore $\phi(f) \in \text{Hom}_c(G/H, \gamma_n(G))$ and hence $\phi(\text{Aut}_c^{n-1}(G)) \subseteq \text{Hom}_c(G/H, \gamma_n(G))$.

Let $\psi \in \text{Hom}_c(G/H, \gamma_n(G))$ be arbitrary. We define the map $f_\psi : G \rightarrow G$ by $f_\psi(g) = g\psi(gH)$ for all $g \in G$. Then by using the argument similar to Theorem 3.3(2), $f_\psi \in \text{Aut}_H^{\gamma_n(G)}(G)$. Now it is easy to see that $f_\psi \in \text{Aut}_c^{n-1}(G)$ and $\phi(f_\psi) = \alpha_{f_\psi} = \psi$. Thus $\text{Hom}_c(G/H, \gamma_n(G)) \subseteq \phi(\text{Aut}_c^{n-1}(G))$. Hence the map ϕ is an isomorphism from $\text{Aut}_c^{n-1}(G)$ to $\text{Hom}_c(G/H, \gamma_n(G))$. \square

Corollary 1. *Let G be a finite nilpotent group of class n . Then $\text{Aut}_c^{n-1}(G)$ is isomorphic to $\text{Hom}_c(G/Z(G), \gamma_n(G))$.*

Proof. Take $H = Z(G)$. Then the result follows from Theorem 3.4. \square

Now the result of Yadav [8, Proposition 3.1] follows from Corollary 1 by putting $n = 2$.

4. P-GROUPS WITH $\text{Autcent}(G)$ EQUAL TO EITHER $\text{Aut}_c^{n-1}(G)$ OR $\text{Aut}_{Z(G)}^{\gamma_n(G)}(G)$

Here we discuss the equality of central automorphisms with $(n-1)^{\text{th}}$ class preserving automorphisms and with $\text{Aut}_{Z(G)}^{\gamma_n(G)}(G)$.

Theorem 4.1. *Let G be a finite non-abelian p -group and $n \geq 2$. Let $\text{Aut}_c^{n-1}(G) = \text{Autcent}(G)$. Then*

- (1) $\gamma_n(G) \leq Z(G)$ and $d(\gamma_n(G)) = d(Z(G))$.
- (2) If $\exp(G/\gamma_2(G)) > \text{var}(\gamma_n(G), Z(G))$, then $\gamma_n(G) = Z(G)$ and $\text{Aut}_c^{n-1}(G)$ is isomorphic to $\text{Hom}(G/Z(G), \gamma_n(G))$.

Conversely, if $\gamma_n(G) = Z(G)$ and $\text{Aut}_c^{n-1}(G)$ is isomorphic to $\text{Hom}(G/Z(G), \gamma_n(G))$, then $\text{Aut}_c^{n-1}(G) = \text{Autcent}(G)$ and $\exp(G/\gamma_2(G)) > \text{var}(\gamma_n(G), Z(G))$.

Proof. (1) Let $\text{Aut}_c^{n-1}(G) = \text{Autcent}(G)$. We first prove that $\gamma_n(G) \leq Z(G)$. Let $g \in G$ and $x \in \gamma_{n-1}(G)$. Then $[g, x] = g^{-1}\sigma_{x^{-1}}(g)$, where $\sigma_{x^{-1}}$ is defined as $\sigma_{x^{-1}}(a) = x^{-1}ax$ for all $a \in G$. As $\sigma_{x^{-1}} \in \text{Aut}_c^{n-1}(G) = \text{Autcent}(G)$, we get $[g, x] = g^{-1}\sigma_{x^{-1}}(g) \in Z(G)$, and therefore $\gamma_n(G) \leq Z(G)$.

To prove that $d(\gamma_n(G)) = d(Z(G))$, first we show that G is purely non-abelian. Suppose, if possible, G has a non-trivial abelian direct factor, say $G = H \times A$ where H is a purely non-abelian subgroup and A is a non-trivial abelian direct factor of G . Note that $\gamma_n(H) \leq Z(H)$. Then using Lemma 4 and Theorem 3.2, $|\text{Autcent}(G)| > |\text{Autcent}(H)| \geq |\text{Aut}_c^{n-1}(H)| = |\text{Aut}_c^{n-1}(G)| = |\text{Autcent}(G)|$, which is absurd. Thus G is purely non-abelian. Hence, by Theorem 2.1,

$$|\text{Autcent}(G)| = |\text{Hom}(G/\gamma_2(G), Z(G))|.$$

Now for $f \in \text{Aut}_c^{n-1}(G)$, by Theorem 3.3(1), if we take $M = \gamma_n(G)$ and $N = \{1\}$, the map $\alpha_f \in \text{Hom}(G, \gamma_n(G))$. As $\gamma_2(G) \subseteq \ker(\alpha_f)$; α_f can be extended to a homomorphism from $G/\gamma_2(G)$ to $\gamma_n(G)$. Therefore, $|\text{Aut}_c^{n-1}(G)| \leq |\text{Hom}(G/\gamma_2(G), \gamma_n(G))|$. Thus by the given condition,

$$\text{Hom}(G/\gamma_2(G), Z(G)) = \text{Hom}(G/\gamma_2(G), \gamma_n(G)).$$

Then by using Lemma 5, $d(\gamma_n(G)) = d(Z(G))$.

- (2) Since $\exp(G/\gamma_2(G)) > \text{var}(\gamma_n(G), Z(G))$, therefore by using Lemma 5, we get $\gamma_n(G) = Z(G)$.

Note that $\text{Autcent}(G) = \text{Aut}_{\gamma_n(G)}^{Z(G)}(G)$. Then by Theorem 3.3(2), if we take $M = Z(G)$ and $N = \gamma_n(G)$, $\text{Autcent}(G)$ is isomorphic to

$Hom(G/\gamma_n(G), Z(G))$. Since $\gamma_n(G) = Z(G)$, therefore $Autcent(G)$ is isomorphic to $Hom(G/Z(G), \gamma_n(G))$. As $Aut_c^{n-1}(G) = Autcent(G)$, it follows that $Aut_c^{n-1}(G)$ is isomorphic to $Hom(G/Z(G), \gamma_n(G))$.

Conversely, assume that $\gamma_n(G) = Z(G)$ and $Aut_c^{n-1}(G)$ is isomorphic to $Hom(G/Z(G), \gamma_n(G))$. Now as $\gamma_n(G) = Z(G)$, therefore G is purely non-abelian and

$Aut_c^{n-1}(G) \subseteq Autcent(G)$. Hence, by using Theorem 2.1, we get,

$$\begin{aligned} |Autcent(G)| &= |Hom(G/\gamma_n(G), Z(G))| \\ &= |Hom(G/Z(G), \gamma_n(G))| \\ &= |Aut_c^{n-1}(G)|. \end{aligned}$$

Therefore, $Autcent(G) = Aut_c^{n-1}(G)$. Clearly, $exp(G/\gamma_2(G)) > var(\gamma_n(G), Z(G))$. \square

Now we prove the result of Kalra and Gumber [6, Theorem 3.1].

Corollary 2. [6, Theorem 3.1] *Let G be a finite non-abelian p -group. Then $Aut_c(G) = Autcent(G)$ if and only if $Aut_c(G)$ is isomorphic to $Hom(G/Z(G), \gamma_2(G))$ and $\gamma_2(G) = Z(G)$.*

Proof. Note that $var(\gamma_2(G), Z(G)) < exp(\gamma_2(G)) = exp(G/Z(G)) \leq exp(G/\gamma_2(G))$. Now the result follows from Theorem 4.1. \square

Lemma 7. *Let $n \geq 2$ and let G be a finite non-abelian p -group such that $Aut_{Z(G)}^{\gamma_n(G)}(G) = Autcent(G)$. Then G is purely non-abelian and $\gamma_n(G) \leq Z(G)$.*

Proof. Let us assume that $Aut_{Z(G)}^{\gamma_n(G)}(G) = Autcent(G)$. Then by the similar argument as in Theorem 4.1, we get $\gamma_n(G) \leq Z(G)$. Next, we show that G is purely non-abelian. Suppose G is not purely non-abelian, say $G = H \times A$ where H is a purely non-abelian subgroup and A is a non-trivial abelian direct factor of G . Note that $\gamma_n(H) \leq Z(H)$. Then by using Lemma 4 and Theorem 3.2, we get $|Autcent(G)| > |Autcent(H)| \geq |Aut_{Z(H)}^{\gamma_n(H)}(H)| = |Aut_{Z(G)}^{\gamma_n(G)}(G)| = |Autcent(G)|$, which is absurd. Hence G is purely non-abelian. \square

The following theorem is a particular case of [2, Corollary C]. Here we have provided a different proof.

Theorem 4.2. *Let G be a finite non-abelian p -group and let $n \geq 2$. Let $Aut_{Z(G)}^{\gamma_n(G)}(G) = Autcent(G)$. Then*

- (1) $\gamma_n(G) \leq Z(G)$ and $d(\gamma_n(G)) = d(Z(G))$.
- (2) If $exp(G/\gamma_2(G)) > var(\gamma_n(G), Z(G))$, then $\gamma_n(G) = Z(G)$.

Conversely, if $\gamma_n(G) = Z(G)$, then $exp(G/\gamma_2(G)) > var(\gamma_n(G), Z(G))$ and $Aut_{Z(G)}^{\gamma_n(G)}(G) = Autcent(G)$

Proof. (1) Let us assume that $Aut_{Z(G)}^{\gamma_n(G)}(G) = Autcent(G)$. Then by Lemma 7, G is purely non-abelian and $\gamma_n(G) \leq Z(G)$. As G is purely non-abelian, by Theorem 2.1, we get $|Autcent(G)| = |Hom(G/\gamma_2(G), Z(G))|$.

Now for $f \in Aut_{Z(G)}^{\gamma_n(G)}(G) \subseteq Aut^{\gamma_n(G)}(G)$, by Theorem 3.3(1), if we take $M = \gamma_n(G)$ and $N = \{1\}$, the map $\alpha_f \in Hom(G, \gamma_n(G))$. As $\gamma_2(G) \subseteq ker(\alpha_f)$; α_f can be extended to a homomorphism from $G/\gamma_2(G)$ to $\gamma_n(G)$. Therefore, $|Aut_{Z(G)}^{\gamma_n(G)}(G)| \leq |Hom(G/\gamma_2(G), \gamma_n(G))|$. Thus we have,

$$\begin{aligned} |Hom(G/\gamma_2(G), Z(G))| &= |Autcent(G)| \\ &= |Aut_{Z(G)}^{\gamma_n(G)}(G)| \\ &\leq |Hom(G/\gamma_2(G), \gamma_n(G))|. \end{aligned}$$

Hence, $Hom(G/\gamma_2(G), Z(G)) = Hom(G/\gamma_2(G), \gamma_n(G))$. Therefore by using Lemma 5, $d(\gamma_n(G)) = d(Z(G))$.

(2) Since $exp(G/\gamma_2(G)) > var(\gamma_n(G), Z(G))$, therefore by Lemma 5, we get $\gamma_n(G) = Z(G)$.

Conversely, assume that $\gamma_n(G) = Z(G)$. Then we have

$$Autcent(G) = Aut_{\gamma_2(G)}^{Z(G)}(G) \subseteq Aut_{\gamma_n(G)}^{Z(G)}(G) = Aut_{Z(G)}^{\gamma_n(G)}(G).$$

Thus $Autcent(G) \subseteq Aut_{Z(G)}^{\gamma_n(G)}(G)$. Hence $Aut_{Z(G)}^{\gamma_n(G)}(G) = Autcent(G)$.

Since $\gamma_n(G) = Z(G)$, $exp(G/\gamma_2(G)) > var(\gamma_n(G), Z(G))$. □

The result of Rai [7, Theorem B(1)] follows from the following corollary.

Corollary 3. *Let G be a finite non-abelian p -group. Then $Aut_{Z(G)}^{\gamma_2(G)}(G) = Autcent(G)$ if and only if $\gamma_2(G) = Z(G)$.*

Proof. Note that $var(\gamma_2(G), Z(G)) < exp(\gamma_2(G)) = exp(G/Z(G)) \leq exp(G/\gamma_2(G))$. Now the proof follows from Theorem 4.2. □

ACKNOWLEDGEMENT

I would like to thank the referee for his/her valuable comments, specially for suggesting Lemma 6 and Theorem 3.3 for better presentation of the article.

REFERENCES

- [1] Adney, J. E. and Yen, T., Automorphisms of a p -group, Illinois J. Math. 9 (1965) 137-143.
- [2] Azhdari, Z. and Malayeri M. A., On automorphisms fixing certain groups, J. Algebra Appl. 12 (2013) 1250163.
- [3] Bidwell, J. N. S., Curran, M. J. and McCaughan, D. J., Automorphisms of direct products of finite groups, Arch. Math.(Basel) 86 (2006) 481-489.
- [4] Dummit, D. S. and Foote, R. M., *Abstract Algebra*, John Wiley and Sons, 2004.

- [5] Gumber, D. and Kalra, H., Equality of certain automorphism groups of finite p -groups, arXiv preprint arXiv:1406.0237 (2014).
- [6] Kalra, H. and Gumber D., On equality of central and class preserving automorphisms of finite p -groups, Indian J. Pure Appl. Math. 44 (2013) 711-725.
- [7] Rai, P. K., On IA-automorphisms that fix the center element-wise, Proc. Indian Acad. Sci. Math. Sci. 124 (2014) 169-173.
- [8] Yadav, M. K., On automorphisms of some finite p -groups, Proc. Indian acad. Sci. Math. Sci. 118 (2008) 1-11.
- [9] Yadav, M. K., On central automorphisms fixing the center element-wise, Comm. Algebra 37 (2009) 4325-4331.

DISCIPLINE OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY, GANDHINAGAR 382424, INDIA.

E-mail address: `surjeetkour@iitgn.ac.in`, `vishakha.maths@iitgn.ac.in`